Hilbert Spaces 2026 (MATS2210)

Assignment 7

Solutions

Exercise 7.1

Recall
$$\delta(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}.$$

For any $\lambda \in \mathbb{C}$,
$$(T - \lambda I)^* = T^* - \overline{\lambda} I.$$
Moreover, If $T \in L(H)$ is invertible if and only if T^* is invertible, we have the following.
$$(T^{-1})^* = (T^*)^{-1}.$$

Observe that
$$T - \lambda I \text{ is invertible}$$
$\Longrightarrow$
$$(T - \lambda I)^* \text{ is invertible}$$
$\Longrightarrow$
$$T^* - \overline{\lambda} I \text{ is invertible.}$$
and vice versa.

Therefore, we get
$$T - \lambda I \text{ is not invertible} \iff T^* - \overline{\lambda} I \text{ is not invertible.}$$

Therefore
$$\lambda \in \delta(T) \iff \overline{\lambda} \in \delta(T^*),$$
and hence
$$\delta(T^*) = \{\overline{\lambda} : \lambda \in \delta(T)\}.$$

Exercise 7.2

From exercise 7.1, we know that

$$(T-\lambda I)^* = T^* - \overline{\lambda} I,$$

since $(\lambda I)^* = \overline{\lambda} I$.

Now consider

$$(T-\lambda I)(T-\lambda I)^* = (T-\lambda I)(T^* - \overline{\lambda} I) = TT^* - \overline{\lambda} T - \lambda T^* + |\lambda|^2 I. \tag{a}$$

and,

$$(T-\lambda I)^*(T-\lambda I) = (T^* - \overline{\lambda} I)(T-\lambda I) = T^*T - \lambda T^* - \overline{\lambda} T + |\lambda|^2 I. \tag{b}$$

Since T is normal, we have $TT^* = T^*T$. From (a) and (b), we conclude that

$$(T-\lambda I)(T-\lambda I)^* = (T-\lambda I)^*(T-\lambda I).$$

Therefore, $T - \lambda I$ is normal.

[Note: It is not required to assume $\lambda \in \delta(T)$.]

Exercise 7.3

Let us define

$$S := T - \lambda I.$$

Since T is normal, the operator S is also normal. Since $\lambda \in \delta(T)$ implies that $0 \in \delta(S)$, so S is not invertible.

Now, on contrary assume that no such sequence exists. Then there exists a constant $c > 0$ such that

$$\|Sx\| \geq c\|x\| \quad \text{for all } x \in H.$$

This implies that S is injective and that $\mathrm{Ran}(S)$ is closed. Since S is normal, we have

$$\ker(S) = \ker(S^*),$$

Let us recall that

$$\mathrm{Ran}(S)^\perp = \ker(S^*) \qquad \text{for all } S \in L(H).$$

and,

$$\overline{\mathrm{Ran}(S)} = (\ker S^*)^\perp.$$

Since $\mathrm{Ran}(S)$ is closed, we obtain

$$\mathrm{Ran}(S) = (\ker S^*)^\perp.$$

Finally, because S is normal, we have

$$\ker(S) = \ker(S^*).$$

Thus

$$\operatorname{Ran}(S) = (\ker S^*)^\perp = (\ker S)^\perp.$$

and therefore

$$\overline{\operatorname{Ran}(S)} = (\ker S^*)^\perp = (\ker S)^\perp.$$

Because S is injective, $\ker(S) = \{0\}$, hence

$$\overline{\operatorname{Ran}(S)} = H.$$

Since the range is closed, this gives $\operatorname{Ran}(S) = H$.

Thus S is bijective. By the bounded inverse theorem, S has a bounded inverse, so S is invertible. This contradicts $0 \in \delta(S)$.

Therefore, our assumption was not correct. Hence for every $n \in \mathbb{N}$, there exists $x_n \in H$ with $\|x_n\| = 1$ such that

$$\|Sx_n\| < \frac{1}{n}.$$

Since $S = T - \lambda I$, this means

$$\|(T - \lambda I)x_n\| \to 0.$$

□

Exercise 7.4

Let ϕ be defined for a polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$ by

$$\phi(p) = p(T) := \sum_{k=0}^{n} a_k T^k.$$

This is well-defined since $T \in L(H)$ and powers of bounded operators are again bounded.

- Let $p(z) = \sum_{k=0}^{n} a_k z^k$ and $q(z) = \sum_{j=0}^{m} b_j z^j$. Then

$$p(T)q(T) = \left(\sum_{k=0}^{n} a_k T^k\right)\left(\sum_{j=0}^{m} b_j T^j\right) = \sum_{k,j} a_k b_j T^{k+j}.$$

  Since

$$(pq)(z) = \sum_{k,j} a_k b_j z^{k+j},$$

  we obtain

$$p(T)q(T) = (pq)(T).$$

  Thus

$$\phi(pq) = \phi(p)\phi(q).$$

- For $\lambda \in \mathbb{C}$,

$$\phi(\lambda p) = (\lambda p)(T) = \sum \lambda a_k T^k = \lambda \sum a_k T^k = \lambda p(T) = \lambda \phi(p).$$

- For the constant polynomial 1,

$$\phi(1) = 1(T) = I.$$

- Let $\overline{p}(z) = \sum \overline{a_k} z^k$. Then

$$p(T)^* = \left(\sum a_k T^k\right)^* = \sum \overline{a_k} (T^k)^*.$$

  Since T is self-adjoint, $(T^k)^* = T^k$. Hence

$$p(T)^* = \sum \overline{a_k} T^k = \overline{p}(T) = \phi(\overline{p}).$$

- If $f(z) = z$, then

$$\phi(f) = f(T) = T.$$

- $\|\phi(p)\| = \|p\|_\infty$:
  Since T is self-adjoint (hence normal), every polynomial $p(T)$ is also normal. For a normal operator S we have

$$\|S\| = \sup_{\lambda \in \delta(S)} |\lambda|.$$

  By theorem 7.5 in lecture notes,

$$\delta(p(T)) = p(\delta(T)).$$

  Therefore

$$\|p(T)\| = \sup_{\lambda \in \delta(p(T))} |\lambda| = \sup_{\mu \in \delta(T)} |p(\mu)| = \|p\|_\infty.$$

Thus the map ϕ exists and satisfies all the required properties.