

Hilbert Spaces 2026 (MATS2210)

Assignment 6

Solutions

Exercise 6.1

Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed spaces, and let $\Lambda : E \rightarrow F$ be an additive isometry. Assume that

$$\Lambda(x) = \Lambda(y)$$

for some $x, y \in E$. Since Λ is additive, we have

$$\Lambda(x) - \Lambda(y) = 0,$$

and hence

$$\Lambda(x - y) = 0.$$

By isometry,

$$\|\Lambda(x - y)\|_F = \|x - y\|_E.$$

since $\Lambda(x - y) = 0$, it follows that

$$\|x - y\|_E = \|\Lambda(x - y)\|_F = 0.$$

Therefore, $\|x - y\|_E = 0$ implies that $x - y = 0$ implies that $x = y$.

Thus, Λ is injective.

Exercise 6.2

Let

$$T : \ell^1 \rightarrow c_0$$

be defined by

$$T(x) = \left(\sum_{j=1}^{\infty} x_j, \sum_{j=2}^{\infty} x_j, \sum_{j=3}^{\infty} x_j, \dots \right),$$

where $x = (x_j)_{j \in \mathbb{N}} \in \ell^1$.

- **Well-defined:**

Since $x \in \ell^1$, we have

$$\sum_{j=1}^{\infty} |x_j| < \infty.$$

Define

$$(Tx)_k = \sum_{j=k}^{\infty} x_j.$$

Since the series is absolutely convergent, we get

$$\lim_{k \rightarrow \infty} \sum_{j=k}^{\infty} x_j = 0.$$

Hence,

$$(Tx)_k \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

which shows that $T(x) \in c_0$. Thus, T is well-defined.

- **Linearity:**

Let $x, y \in \ell^1$ and $\alpha \in \mathbb{C}$.

$$T(\alpha x + y)_k = \sum_{j=k}^{\infty} (\alpha x_j + y_j) = \sum_{j=k}^{\infty} \alpha x_j + \sum_{j=k}^{\infty} y_j = \alpha(Tx)_k + (Ty)_k.$$

Thus, T is linear.

- **Continuity:**

For each k ,

$$|(Tx)_k| = \left| \sum_{j=k}^{\infty} x_j \right| \leq \sum_{j=k}^{\infty} |x_j| \leq \sum_{j=1}^{\infty} |x_j| = \|x\|_{\ell^1}.$$

Taking the supremum over k , we obtain

$$\|Tx\|_{c_0} = \sup_k |(Tx)_k| \leq \|x\|_{\ell^1}.$$

Hence,

$$\|T\| \leq 1.$$

Therefore, T is bounded and thus continuous.

Hence, $T : \ell^1 \rightarrow c_0$ is linear and continuous.

Exercise 6.3

Recall that

$$T : \ell^1 \rightarrow c_0, \quad (Tx)_k = \sum_{j=k}^{\infty} x_j.$$

Since

$$(c_0)^* = \ell^1 \quad \text{and} \quad (\ell^1)^* = \ell^\infty,$$

the adjoint operator satisfies

$$T^* : \ell^1 \rightarrow \ell^\infty.$$

Let $y^* = (y_k)_{k \in \mathbb{N}} \in \ell^1$ and $x = (x_j)_{j \in \mathbb{N}} \in \ell^1$. By definition of the adjoint,

$$\langle T^* y^*, x \rangle = \langle y^*, Tx \rangle.$$

We compute

$$\langle y^*, Tx \rangle = \sum_{k=1}^{\infty} y_k (Tx)_k = \sum_{k=1}^{\infty} y_k \sum_{j=k}^{\infty} x_j.$$

Due to absolute convergence, we may interchange the order of summation:

$$= \sum_{j=1}^{\infty} x_j \sum_{k=1}^j y_k.$$

Hence

$$\langle y^*, Tx \rangle = \sum_{j=1}^{\infty} \left(\sum_{k=1}^j y_k \right) x_j.$$

Comparing this with

$$\langle T^* y^*, x \rangle = \sum_{j=1}^{\infty} (T^* y^*)_j x_j,$$

we obtain

$$(T^* y^*)_j = \sum_{k=1}^j y_k.$$

Now, since $y^* \in \ell^1$,

$$\left| \sum_{k=1}^n y_k \right| \leq \sum_{k=1}^{\infty} |y_k| = \|y^*\|_{\ell^1},$$

so the sequence $(\sum_{k=1}^n y_k)_{n \in \mathbb{N}}$ is bounded. Thus

$$T^* y^* \in \ell^\infty$$

and

$$T^* y^* = \left(\sum_{k=1}^n y_k \right)_{n \in \mathbb{N}}.$$

□

Exercise 6.4

Let H be a Hilbert space and $T \in L(H)$.

- **If $|\lambda| > \|T\|$, then $\lambda \notin \delta(T)$:**

Assume $|\lambda| > \|T\|$. We write

$$T - \lambda I = -\lambda \left(I - \frac{1}{\lambda} T \right).$$

Then

$$\left\| \frac{1}{\lambda} T \right\| = \frac{\|T\|}{|\lambda|} < 1.$$

Hence, by the Neumann series,

$$\left(I - \frac{1}{\lambda} T \right)^{-1} = \sum_{n=0}^{\infty} \left(\frac{1}{\lambda} T \right)^n,$$

so $I - \frac{1}{\lambda} T$ is invertible. Consequently, $T - \lambda I$ is invertible, and therefore

$$\lambda \notin \delta(T).$$

Thus

$$\delta(T) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\}.$$

- **The spectrum $\delta(T)$ is compact and nonempty:**

We observe from above part, $\delta(T)$ is contained in the closed disk $\{|\lambda| \leq \|T\|\}$, hence it is bounded.

Now, let $\lambda \in \rho(T)$ and denote

$$R := (T - \lambda I)^{-1}.$$

Then $R \in L(H)$.

If

$$|\tilde{\lambda} - \lambda| < \frac{1}{\|R\|},$$

then

$$\|(\lambda - \tilde{\lambda})R\| < 1.$$

Hence the Neumann series

$$\sum_{m=0}^{\infty} (\lambda - \tilde{\lambda})^m R^{m+1}$$

converges in operator norm.

Define

$$\tilde{R} := \sum_{m=0}^{\infty} (\lambda - \tilde{\lambda})^m R^{m+1}.$$

We compute:

$$(T - \tilde{\lambda}I) = (T - \lambda I) + (\lambda - \tilde{\lambda})I.$$

Then

$$(T - \tilde{\lambda}I)\tilde{R} = \tilde{R}(T - \lambda I) + (\lambda - \tilde{\lambda})\tilde{R}.$$

Using the series expansion, one obtains

$$(T - \tilde{\lambda}I)\tilde{R} = I, \quad \tilde{R}(T - \tilde{\lambda}I) = I.$$

Thus

$$\tilde{R} = (T - \tilde{\lambda}I)^{-1},$$

so $\tilde{\lambda} \in \rho(T)$.

Therefore $\rho(T)$ is open in \mathbb{C} .

Since $\delta(T) = \mathbb{C} \setminus \rho(T)$, it follows that

$\delta(T)$ is closed.

Now we discuss the non-emptiness of $\delta(T)$. On the contrary, assume that $\delta(T) = \emptyset$. Then for all $\lambda \in \mathbb{C}$, the operator $T - \lambda I$ is invertible. Define the *resolvent function*:

$$R(\lambda) := (T - \lambda I)^{-1} \in L(H).$$

Note that $R(\lambda)$ is holomorphic on \mathbb{C} with values in $L(H)$.

Now let us see the behavior of R towards infinity.

For $|\lambda| > \|T\|$, we have

$$R(\lambda) = (T - \lambda I)^{-1} = -\frac{1}{\lambda} \left(I - \frac{T}{\lambda} \right)^{-1} = -\frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{T}{\lambda} \right)^n,$$

so that

$$\|R(\lambda)\| \leq \frac{1}{|\lambda| - \|T\|} \rightarrow 0 \quad \text{as } |\lambda| \rightarrow \infty.$$

Therefore, by Liouville's theorem, a bounded entire function with values in a Banach space that tends to 0 at infinity must be identically 0. Hence $R(\lambda) \equiv 0$, which is a contradiction as 0 is not invertible.

Hence,

$$\delta(T) \neq \emptyset.$$

□