Hilbert Spaces 2026 (MATS2210)

Assignment 5

Solutions

Exercise 5.1

Let $f, g : [0, 2\pi] \to \mathbb{C}$ be continuous 2π-periodic functions. Recall the definition of

$$(g \star f)(x) = \frac{1}{2\pi} \int_0^{2\pi} f(t) g(x - t)\, dt.$$

Make the substitution

$$u = x - t.$$

Then

$$t = x - u, \qquad dt = -du.$$

When $t = 0$, $u = x$. When $t = 2\pi$, $u = x - 2\pi$.
Thus,

$$(g \star f)(x) = \frac{1}{2\pi} \int_x^{x-2\pi} f(x - u) g(u) (-du).$$

Reversing the limits:

$$= \frac{1}{2\pi} \int_{x-2\pi}^{x} f(x - u) g(u)\, du.$$

Since f and g are 2π-periodic,

$$\int_{x-2\pi}^{x} = \int_0^{2\pi}.$$

Therefore,

$$(g \star f)(x) = \frac{1}{2\pi} \int_0^{2\pi} g(u) f(x - u)\, du = (f \star g)(x).$$

Hence, $g \star f = f \star g$.

Exercise 5.2

Let $D_n(x)$ be the n-th Dirichlet kernel:

$$D_n(x) = \sum_{k=-n}^{n} e^{ikx}.$$

We can rewrite it as a geometric series:

$$\sum_{k=-n}^{n} e^{ikx} = e^{-inx} \sum_{k=0}^{2n} e^{ikx} = e^{-inx} \frac{1 - e^{i(2n+1)x}}{1 - e^{ix}}.$$

Factor $e^{ix/2}$ in numerator and denominator:

$$D_n(x) = e^{-inx} \frac{e^{i(2n+1)x/2} - e^{-i(2n+1)x/2}}{e^{ix/2} - e^{-ix/2}} = \frac{\sin((2n+1)x/2)}{\sin(x/2)} = \frac{\sin((n + \frac{1}{2})x)}{\sin(x/2)}.$$

Hence,

$$D_n(x) = \frac{\sin((n + \frac{1}{2})x)}{\sin(x/2)}.$$

Exercise 5.3

Let $f, g \in C([0, 2\pi])$ be 2π-periodic. The convolution is defined as

$$(f * g)(x) := \frac{1}{2\pi} \int_0^{2\pi} f(t) g(x - t) \, dt.$$

Consider

$$(f * g)^\wedge(k) = \frac{1}{2\pi} \int_0^{2\pi} (f * g)(x) e^{-ikx} dx = \frac{1}{2\pi} \int_0^{2\pi} \left[\frac{1}{2\pi} \int_0^{2\pi} f(t) g(x - t) dt \right] e^{-ikx} dx.$$

Apply Fubini to change the order of integration:

$$(f * g)^\wedge(k) = \frac{1}{(2\pi)^2} \int_0^{2\pi} f(t) \left[\int_0^{2\pi} g(x - t) e^{-ikx} dx \right] dt.$$

Now, let $u = x - t$

$$\int_0^{2\pi} g(x - t) e^{-ikx} dx = \int_0^{2\pi} g(u) e^{-ik(u+t)} du = e^{-ikt} \int_0^{2\pi} g(u) e^{-iku} du.$$

We obtain

$$(f * g)^\wedge(k) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt} dt \cdot \frac{1}{2\pi} \int_0^{2\pi} g(u) e^{-iku} du = \hat{f}(k) \, \hat{g}(k).$$

Therefore,

$$(f * g)^\wedge(k) = \hat{f}(k) \hat{g}(k)$$

Exercise 5.4

Let E be a Banach space, and define

$$\mathrm{Inv}(L(E)) := \{T \in L(E) : T \text{ is invertible}\}.$$

(a) Inv$(L(E))$ is a group: We will verify the axioms of group.

- **Closure:** If $T_1, T_2 \in \mathrm{Inv}(L(E))$, then

$$(T_1T_2)^{-1} = T_2^{-1}T_1^{-1} \in \mathrm{Inv}(L(E)).$$

- **Associativity:** It follows from the composition of linear operators.
- **Identity:** The identity I satisfies $I^{-1} = I$, so $I \in \mathrm{Inv}(L(E))$.
- **Inverse:** By definition, each $T \in \mathrm{Inv}(L(E))$ has an inverse $T^{-1} \in L(E)$.

Hence, $\mathrm{Inv}(L(E))$ is a group under composition.

(b) Inv$(L(E))$ is open:
Let $T \in \mathrm{Inv}(L(E))$ and define $\epsilon := \frac{1}{\|T^{-1}\|}$. If $S \in L(E)$ satisfies $\|S - T\| < \epsilon$, set

$$R := T - S, \quad \text{so that } \|T^{-1}R\| < 1.$$

Then

$$S = T - R = T(I - T^{-1}R).$$

Since $\|T^{-1}R\| < 1$, the Neumann series

$$(I - T^{-1}R)^{-1} = \sum_{n=0}^{\infty} (T^{-1}R)^n$$

converges in $L(E)$, so $I - T^{-1}R$ is invertible. Therefore,

$$S^{-1} = (I - T^{-1}R)^{-1}T^{-1} \in L(E),$$

and $S \in \mathrm{Inv}(L(E))$.

Therefore we get

$$\{S \in L(E) : \|S - T\| < \epsilon\} \subset \mathrm{Inv}(L(E)),$$

and that $\mathrm{Inv}(L(E))$ is open in $(L(E), \|\cdot\|)$.