Hilbert Spaces 2026 (MATS2210)

Assignment 4

Solutions

Exercise 4.1

(a) The inner product on $L^2([0, 2\pi])$:
We verify the inner product axioms.

- **Linearity:** For $x_1, x_2, y \in L^2([0, 2\pi])$ and $\alpha, \beta \in \mathbb{C}$,

$$(\alpha x_1 + \beta x_2 \mid y) = \int_0^{2\pi} (\alpha x_1(t) + \beta x_2(t))\overline{y(t)}\, dt = \alpha(x_1 \mid y) + \beta(x_2 \mid y).$$

- **Symmetry:**

$$(x \mid y) = \int_0^{2\pi} x(t)\overline{y(t)}\, dt = \overline{\int_0^{2\pi} y(t)\overline{x(t)}\, dt} = \overline{(y \mid x)}.$$

- **Positive definiteness:**

$$(x \mid x) = \int_0^{2\pi} |x(t)|^2\, dt \geq 0,$$

and $(x \mid x) = 0$ if and only if $x(t) = 0$ almost everywhere.

Thus $(\cdot \mid \cdot)$ defines an inner product on $L^2([0, 2\pi])$.

(b) Orthonormality:
For $n, m \in \mathbb{Z}$,

$$(x_n \mid x_m) = \int_0^{2\pi} \frac{e^{int}}{\sqrt{2\pi}} \overline{\frac{e^{imt}}{\sqrt{2\pi}}}\, dt = \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)t}\, dt.$$

If $n = m$, then

$$(x_n \mid x_n) = \frac{1}{2\pi} \int_0^{2\pi} 1\, dt = 1.$$

If $n \neq m$, then

$$\int_0^{2\pi} e^{i(n-m)t}\,dt = \left[\frac{e^{i(n-m)t}}{i(n-m)}\right]_0^{2\pi} = \frac{e^{i(n-m)2\pi}-1}{i(n-m)} = 0,$$

since $e^{i(n-m)2\pi} = 1$.

Therefore,

$$(x_n \mid x_m) = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

Hence,

$$x_n(t) = \frac{e^{int}}{\sqrt{2\pi}}, \quad n \in \mathbb{Z},$$

form an orthonormal sequence in $L^2([0, 2\pi])$. □

Exercise 4.2

We prove the equality by proving both the set inclusions.

- $(\overline{A})^\perp \subset A^\perp$.
  Since $A \subset \overline{A}$, any vector orthogonal to every element of $\overline{A}$ is, in particular, orthogonal to every element of A. Hence,
  $$(\overline{A})^\perp \subset A^\perp.$$

- $A^\perp \subset (\overline{A})^\perp$.
  Let $x \in A^\perp$. Then
  $$\langle x, a\rangle = 0 \quad \text{for all } a \in A.$$
  Let $y \in \overline{A}$. By definition of closure, there exists a sequence $(y_n) \subset A$ such that
  $$y_n \to y \quad \text{in } E.$$
  Since the inner product is continuous,
  $$\langle x, y\rangle = \left\langle x, \lim_{n\to\infty} y_n \right\rangle = \lim_{n\to\infty} \langle x, y_n\rangle.$$
  But $\langle x, y_n\rangle = 0$ for all n, hence
  $$\langle x, y\rangle = 0.$$
  Therefore
  $$x \in (\overline{A})^\perp.$$

Hence,

$$(\overline{A})^\perp = A^\perp.$$

□

Exercise 4.3

Let $(e_n)_{n\in\mathbb{N}}$ be a countable Hilbert basis of the Hilbert space E.

Define
$$D := \left\{ \sum_{k=1}^{n} q_k e_k \;\middle|\; n \in \mathbb{N},\ q_k \in \mathbb{Q} \right\}.$$
Observe that the set D is countable, since it is a countable union of finite products of $\mathbb{Q}$, which are countable.

Now to show D is dense in E. Let $x \in E$. Since (e_n) is a Hilbert basis, x has the expansion
$$x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n,$$
with convergence in the norm of E.

For any $\varepsilon > 0$, choose N such that
$$\left\| x - \sum_{n=1}^{N} \langle x, e_n \rangle e_n \right\| < \frac{\varepsilon}{2}.$$
Next, for each $1 \le n \le N$, choose a rational number q_n such that
$$|\langle x, e_n \rangle - q_n| < \frac{\varepsilon}{2\sqrt{N}}.$$
Define
$$y := \sum_{n=1}^{N} q_n e_n \in D.$$
Then, using orthonormality and parseval identity i.e., for every $x \in E$, we have
$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2.$$
where
$$x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n.$$
We obtain
$$\|x - y\|^2 = \sum_{n=1}^{N} |\langle x, e_n \rangle - q_n|^2 + \sum_{n=N+1}^{\infty} |\langle x, e_n \rangle|^2.$$
By the choice of N and q_n, we have
$$\|x - y\| < \varepsilon.$$
Since every element of E can be approximated by the elements of D.
Therefore D is dense in E and E is separable.

$\square$

Exercise 4.4

Recall that the orthogonal projection $P_M : E \to M$ is defined by

$$P_M(x) := u.$$

where

$$x = u + v,$$

for $u \in M$ and $v \in M^\perp$.

We need to verify

$$P_M(\alpha x + \beta y) = \alpha P_M(x) + \beta P_M(y)$$

for all $x, y \in E$ and scalars $\alpha, \beta \in \mathbb{R}$.

Now represent x and y

$$x = u_x + v_x, \quad u_x \in M,\ v_x \in M^\perp,$$

$$y = u_y + v_y, \quad u_y \in M,\ v_y \in M^\perp.$$

Then for $\alpha, \beta \in \mathbb{R}$,

$$\alpha x + \beta y = \alpha u_x + \beta u_y + (\alpha v_x + \beta v_y).$$

Since M and $M^\perp$ are subspace, we get $\alpha u_x + \beta v_y \in M$ and $\alpha v_x + \beta v_y \in M^\perp$.

By uniqueness of the decomposition, the M-component of $\alpha x + \beta y$ is

$$P_M(\alpha x + \beta y) = \alpha u_x + \beta u_y = \alpha P_M(x) + \beta P_M(y).$$

Hence, $P_M : E \to E$ is linear. □