Hilbert Spaces 2026 (MATS2210)

Assignment 3

Solutions

Exercise 3.1

Let $x_0 \in X$ and define the sequence (x_k) by

$$x_{k+1} := T(x_k), \quad k \in \mathbb{N}.$$

We need to show that

$$d(x_m, x_n) \leq \frac{q^n}{1-q} d(x_1, x_0), \quad \text{whenever } m > n.$$

Since T is a contraction, we obtain the following result by induction.

$$d(x_{k+1}, x_k) \leq q^k d(x_1, x_0), \quad k \geq 0.$$

Now let $m > n$. By the triangle inequality,

$$d(x_m, x_n) \leq \sum_{k=n}^{m-1} d(x_{k+1}, x_k).$$

using the above relation and $0 \leq q < 1$. we get

$$d(x_m, x_n) \leq d(x_1, x_0) \sum_{k=n}^{m-1} q^k \leq \frac{q^n}{1-q} d(x_1, x_0).$$

Therefore,

$$d(x_m, x_n) \leq \frac{q^n}{1-q} d(x_1, x_0), \quad \text{for all } m > n.$$

□

Exercise 3.2

We need to show that the sequence (x_k) defined in Exercise 3.1 converges in X, and

$$x^* := \lim_{k\to\infty} x_k$$

where $x^* \in X$ is the unique fixed point of the mapping T that satisfies

$$T(x^*) = x^*.$$

(a) Convergence of (x_k): Because of complete metric space it is enough to show that (x_k) is cauchy. Now from Exercise 3.1 we know that for all $m > n$,

$$d(x_m, x_n) \leq \frac{q^n}{1-q}\, d(x_1, x_0).$$

Since $0 \leq q < 1$, it follows that

$$\frac{q^n}{1-q}\, d(x_1, x_0) \xrightarrow[n\to\infty]{} 0.$$

Hence, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$m, n \geq N \implies d(x_m, x_n) < \varepsilon.$$

Thus, (x_k) is a Cauchy sequence and therefore convergent i.e., there exists $x^* \in X$ such that

$$x_k \xrightarrow[k\to\infty]{} x^*.$$

(b) x^* is a unique fixed point of T.

We will proceed with the hint and first show that T is continuous. Let $x_n \to x$ in X. Since T is a contraction,

$$d(T(x_n), T(x)) \leq q\, d(x_n, x).$$

Now $d(x_n, x) \to 0$, implies that

$$d(T(x_n), T(x)) \to 0,$$

and hence $T(x_n) \to T(x)$. Thus, T is continuous.

We have the sequence, $x_{k+1} = T(x_k)$ for all k. By using the continuity of T and uniqueness of limits, we obtain

$$x_{k+1} \xrightarrow[k\to\infty]{} T(x^*).$$

But also $x_{k+1} \to x^*$, hence

$$T(x^*) = x^*.$$

Now we will show that x^* is unique.

Assume that $y^* \in X$ is another fixed point of T, so that $T(y^*) = y^*$. Then

$$d(x^*, y^*) = d(T(x^*), T(y^*)) \leq q\, d(x^*, y^*).$$

Since $0 \leq q < 1$, this implies

$$(1-q)\, d(x^*, y^*) \leq 0,$$

and therefore

$$d(x^*, y^*) = 0,$$

which yields $x^* = y^*$.

□

Exercise 3.3

We verify the inner product axioms.

Let x, y and $z \in \mathbb{C}^n$, with $\alpha \in \mathbb{C}$.

(a) Conjugate symmetry:

$$(y \mid x) = \sum_{j=1}^{n} y_j \overline{x_j} = \overline{\sum_{j=1}^{n} x_j \overline{y_j}} = \overline{(x \mid y)}.$$

(b) Linearity:

$$(x + z \mid y) = \sum_{j=1}^{n} (x_j + z_j)\overline{y_j} = \sum_{j=1}^{n} x_j \overline{y_j} + \sum_{j=1}^{n} z_j \overline{y_j} = (x \mid y) + (z \mid y),$$

and

$$(\alpha x \mid y) = \sum_{j=1}^{n} \alpha x_j \overline{y_j} = \alpha \sum_{j=1}^{n} x_j \overline{y_j} = \alpha (x \mid y).$$

(c) Positive definiteness:

$$(x \mid x) = \sum_{j=1}^{n} x_j \overline{x_j} = \sum_{j=1}^{n} |x_j|^2 \geq 0.$$

Moreover,

$$(x \mid x) = 0 \iff |x_j|^2 = 0 \text{ for all } j \iff x_j = 0 \text{ for all } j \iff x = 0.$$

Since all inner product axioms are satisfied, $(\cdot \mid \cdot)$ is an inner product on $\mathbb{C}^n$.
The norm induced by an inner product is defined by

$$\|x\| := \sqrt{(x \mid x)}.$$

Hence,

$$\|x\| = \sqrt{\sum_{j=1}^{n} |x_j|^2}, \quad x \in \mathbb{C}^n.$$

This is the standard Euclidean (or ℓ^2) norm on $\mathbb{C}^n$.

□

Exercise 3.4

Let $(E, \|\cdot\|)$ be a real normed space and the norm satisfies the parallelogram identity

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \forall x, y \in E.$$

Define for $x, y \in E$

$$(x|y) := \frac{1}{4}\big(\|x+y\|^2 - \|x-y\|^2\big).$$

We will verify the axioms of inner product.

(a) Symmetry:
Since $\|x+y\| = \|y+x\|$ and $\|x-y\| = \|y-x\|$, we obtain

$$(x|y) = (y|x).$$

(b) Positive definiteness:
Setting $y = x$,

$$(x|x) = \frac{1}{4}(\|2x\|^2 - \|0\|^2) = \frac{1}{4}(4\|x\|^2) = \|x\|^2.$$

Thus $(x|x) \geq 0$, and $(x|x) = 0$ if and only if $x = 0$. Moreover,

$$\|x\| = \sqrt{(x|x)}.$$

(c) linearity:
We prove $(x+z|y) = (x|y) + (z|y)$.
Using the definition,

$$4(x+z|y) = \|x+z+y\|^2 - \|x+z-y\|^2.$$

Apply the parallelogram identity to the pairs $(x+y, z)$ and $(x-y, z)$:

$$\|(x+y)+z\|^2 + \|(x+y)-z\|^2 = 2\|x+y\|^2 + 2\|z\|^2,$$

$$\|(x-y)+z\|^2 + \|(x-y)-z\|^2 = 2\|x-y\|^2 + 2\|z\|^2.$$

Subtracting the second equation from the first gives

$$\|x+y+z\|^2 + \|x+y-z\|^2 - \|x-y+z\|^2 - \|x-y-z\|^2 = 2(\|x+y\|^2 - \|x-y\|^2).$$

Rearranging terms yields

$$4(x+z|y) = 4(x|y) + 4(z|y).$$

Dividing by 4 gives

$$(x+z|y) = (x|y) + (z|y).$$

By repeated additivity,

$$(nx|y) = n(x|y) \quad \text{for } n \in \mathbb{Z}.$$

Now, if $q = \frac{m}{n} \in \mathbb{Q}$, then

$$(qx|y) = \frac{m}{n}(x|y).$$

Lastly, the map $x \mapsto (x|y)$ is continuous. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, homogeneity extends to all $\lambda \in \mathbb{R}$:

$$(\lambda x|y) = \lambda(x|y).$$

Therefore $(\cdot|\cdot)$ is an inner product inducing the norm.