Hilbert Spaces 2026 (MATS2210)

Assignment 2

Solutions

Exercise 2.1

We need to show that there exist constants $C_1, C_2 > 0$ such that

$$C_1\|f\|_\infty \leq \|f\|_1 \leq C_2\|f\|_\infty \quad \text{for all } f \in C^k([0,1]).$$

(a) C_2**:** We have,

$$\|f\|_1 = \sum_{j=0}^{k} \|f^{(j)}\|_\infty.$$

For each j we have

$$\|f^{(j)}\|_\infty \leq \sup_{0\leq l\leq k} \|f^{(l)}\|_\infty = \|f\|_\infty.$$

Since the sum contains $k+1$ terms, it follows that

$$\|f\|_1 \leq (k+1)\|f\|_\infty.$$

(b) C_1**:** We have,

$$\|f\|_\infty = \max_{0\leq j\leq k} \|f^{(j)}\|_\infty.$$

By the property of non-negative real munbers

$$\text{For any real numbers } a_1, a_2, \ldots, a_n \geq 0, \quad \max_{1\leq i\leq n} a_i \leq \sum_{i=1}^{n} a_i.$$

Therefore we get,

$$\|f\|_\infty \leq \sum_{j=0}^{k} \|f^{(j)}\|_\infty = \|f\|_1.$$

Hence by (a) and (b), we obtain

$$\|f\|_\infty \leq \|f\|_1 \leq (k+1)\|f\|_\infty \quad \text{for all } f \in C^k([0,1]).$$

Therefore, the norms $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are equivalent on $C^k([0,1])$.

□

Exercise 2.2

Let $(x_n) \in E$ be a Cauchy sequence and $(y_n) \in E$, a subsequence converges to some $x \in E$.

Claim: (x_n) converges to x.

Proof. Let $\varepsilon > 0$, since (x_n) is Cauchy, $\exists\ N_1 \in \mathbb{N}$ such that

$$\|x_n - x_m\| < \frac{\varepsilon}{2} \quad \text{for all } n, m \geq N_1.$$

Also, (y_n) converges to x, there exists $N_2 \in \mathbb{N}$ such that

$$\|y_n - x\| < \frac{\varepsilon}{2} \quad \text{for all } n \geq N_2.$$

Now choose $N = \max\{N_1, k_{N_2}\}$ then, for all $n \geq \max\{N_1, k_{N_2}\}$, we have

$$\|x_n - x\| \leq \|x_n - y_{k_{N_2}}\| + \|y_{N_2} - x\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|x_n - x\| < \varepsilon \quad \text{for all } n \geq N,$$

which shows that (x_n) converges to x in E. □

Exercise 2.3

We need to show that T is linear and $\exists\ C > 0$ such that

$$\|T(x)\|_{\ell^1} \leq C\,\|x\|_{\ell^2} \quad \text{for all } x \in E.$$

T is clearly linear since for $x = (x_k), y = (y_k) \in \ell^2$ and $\alpha, \beta \in \mathbb{R}$ (or $\mathbb{C}$). We have

$$T(\alpha x + \beta y) = T((\alpha x_k + \beta y_k)_{k\in\mathbb{N}}) = \left(\frac{\alpha x_k + \beta y_k}{k}\right)_{k\in\mathbb{N}} = \alpha T(x) + \beta T(y).$$

Now, let $x = (x_k) \in \ell^2$. Then

$$\|Tx\|_{\ell^1} = \sum_{k=1}^{\infty} \left|\frac{x_k}{k}\right| = \sum_{k=1}^{\infty} |x_k| \cdot \frac{1}{k}.$$

By the Hölder inequality,

$$\sum_{k=1}^{\infty} |x_k| \cdot \frac{1}{k} \leq \left(\sum_{k=1}^{\infty} |x_k|^2\right)^{1/2} \left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right)^{1/2}.$$

From lecture notes we know

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6},$$

Then

$$\|Tx\|_{\ell^1} \leq \left(C = \sqrt{\frac{\pi^2}{6}} \right) \|x\|_{\ell^2}.$$

Thus T is a bounded linear operator from ℓ^2 to ℓ^1, and by definition of operator norm we get,

$$\|T\| \leq \sqrt{\frac{\pi^2}{6}}.$$

□

Exercise 2.4

We will use the following result for any real numbers a, b and $p \geq 1$, we have

$$|a + b|^p \leq 2^p(|a|^p + |b|^p),$$

which follows from the convexity of $t \mapsto |t|^p$.

Now for almost every $x \in \Omega$,

$$|f(x) + g(x)|^p \leq 2^p \left(|f(x)|^p + |g(x)|^p \right).$$

Integrate both side over Ω,

$$\int_\Omega |f + g|^p \, d\mu \leq \int_\Omega 2^p \left(|f|^p + |g|^p \right) \, d\mu = 2^p \left(\int_\Omega |f|^p \, d\mu + \int_\Omega |g|^p \, d\mu \right).$$

Thus we obtain,

$$\int_\Omega |f + g|^p \, d\mu \leq 2^p \left(\|f\|_p^p + \|g\|_p^p \right).$$

□

Exercise 2.5

Let $\mathcal{C}$ be the set of all Cauchy sequences in (X, d). Define an equivalence relation $\sim$ on $\mathcal{C}$ by

$$(x_n) \sim (y_n) \iff \lim_{n \to \infty} d(x_n, y_n) = 0.$$

Let $\overline{X} := \mathcal{C}/\sim$ be the set of equivalence classes, and denote the class of (x_n) by $[x_n]$.

Define $\overline{d} : \overline{X} \times \overline{X} \to \mathbb{R}$ by

$$\overline{d}([x_n],[y_n]) := \lim_{n\to\infty} d(x_n, y_n).$$

This limit exists since (x_n) and (y_n) are Cauchy sequences.

Define an embedding $\iota : X \to \overline{X}$ by

$$\iota(x) := [(x, x, x, \dots)].$$

Then ι is an isometric embedding, since

$$\overline{d}(\iota(x), \iota(y)) = d(x, y).$$

- **X is dense in $\overline{X}$:**

  We will show that for every $[x_n] \in \overline{X}$ and every $\varepsilon > 0$, there exists $x \in X$ such that

  $$\overline{d}([x_n], \iota(x)) < \varepsilon.$$

  Let $[x_n] \in \overline{X}$. Since (x_n) is a Cauchy sequence in X, there exists $N \in \mathbb{N}$ such that

  $$d(x_n, x_m) < \varepsilon \quad \text{for all } n, m \geq N.$$

  Fix $n \geq N$ and consider the constant sequence $(x_n, x_n, x_n, \dots)$. Then

  $$\overline{d}([x_n], \iota(x_n)) = \lim_{k\to\infty} d(x_k, x_n) \leq \varepsilon.$$

  Hence, for every $\varepsilon > 0$, there exists $x_n \in X$ such that

  $$\overline{d}([x_n], \iota(x_n)) < \varepsilon,$$

  which shows that X is dense in $\overline{X}$.

- **Completeness of $\overline{X}$:**

  Let $(a_i)_{i=1}^{\infty} = ([(x_n^i)])_{i=1}^{\infty}$ be a Cauchy sequence in $\overline{X}$. By the density of X in $\overline{X}$, for each $i \in \mathbb{N}$ there exists an index $j_i \in \mathbb{N}$ such that

  $$\overline{d}(a_i, x_{j_i}^i) < \frac{1}{i}.$$

  Since (a_i) is a Cauchy sequence in $\overline{X}$, it follows that the sequence $(x_{j_i}^i)_{i=1}^{\infty}$ is a Cauchy sequence in X. Hence its equivalence class

  $$a_\infty := [(x_{j_i}^i)]$$

  belongs to $\overline{X}$.

  Finally, for each $i \in \mathbb{N}$ we have

  $$\overline{d}(a_i, a_\infty) \leq \overline{d}(a_i, x_{j_i}^i) + \overline{d}(x_{j_i}^i, a_\infty) < \frac{1}{i} + \lim_{k\to\infty} d(x_{j_i}^i, x_{j_k}^k),$$

  which tends to zero as $i \to \infty$. Therefore $a_i \to a_\infty$ in $\overline{X}$.

  Thus every Cauchy sequence in $\overline{X}$ converges, and $(\overline{X}, \overline{d})$ is complete.

□