Hilbert Spaces 2026 (MATS2210)

Assignment 1

Solutions

Exercise 1.1

Let $f, g \in C_c^\infty(\mathbb{R}^n)$ and let $\lambda \in \mathbb{R}$.

- $C_c^\infty(\mathbb{R}^n)$ **is a Vector Space:** The zero function $f(x) = 0$ for all $x \in \mathbb{R}^n$ is smooth and it's support is the empty set, which is compact. Thus, $0 \in C_c^\infty(\mathbb{R}^n)$.
  By using the fact that $\text{supp}(f+g) \subseteq \text{supp}(f) \cup \text{supp}(g)$ and $\text{supp}(\lambda f) = \text{supp}(f)$, we coclude that it is closed under addition and scalar multiplication.

- $(C_c^\infty(\mathbb{R}^n), \|\cdot\|_\infty)$ **is norm space:** Verify the axioms of the norm:

  (1) Non-negativity & Definiteness:

  Since $|f(x)| \geq 0$ for all $x \in \mathbb{R}^n$, then $\max_{x \in \mathbb{R}^n} |f(x)| \geq 0$.
  Now, $\|0\|_\infty = \max_{x \in \mathbb{R}^n} |0| = 0$. Conversely, assume $\|f\|_\infty = 0$. Then:

  $$\max_{x \in \mathbb{R}^n} |f(x)| = 0.$$

  This means that $|f(x)| \leq 0$ for all $x \in \mathbb{R}^n$. Also $|f(x)| \geq 0$ for all $x \in \mathbb{R}^n$.Therefore, $f(x) = 0$ for all $x \in \mathbb{R}^n$., so $f = 0$.

  (2) Homogeneity:

  $$\begin{aligned} \|\lambda f\|_\infty &= \max_{x \in \mathbb{R}^n} |\lambda f(x)| \\ &= \max_{x \in \mathbb{R}^n} (|\lambda| \cdot |f(x)|) \\ &= |\lambda| \max_{x \in \mathbb{R}^n} |f(x)| \\ &= |\lambda| \|f\|_\infty. \end{aligned}$$

  (3) Triangle Inequality:

  Using the triangle inequality of real numbers, we get

  $$||f+g||_\infty = \max_{x \in \mathbb{R}^n} |f(x) + g(x)|$$

$$\le \max_{x\in\mathbb{R}^n}(|f(x)| + |g(x)|)$$
$$\le \max_{x\in\mathbb{R}^n}|f(x)| + \max_{x\in\mathbb{R}^n}|g(x)|$$
$$= ||f||_\infty + ||g||_\infty$$

Therefore, $\|\cdot\|_\infty$ is a norm on $C_c^\infty(\mathbb{R}^n)$ and $(C_c^\infty(\mathbb{R}^n), \|\cdot\|_\infty)$ is a norm space.

Exercise 1.2

Consider $f : \mathbb{R}^n \to \mathbb{R}^n$,

$$f(x) = \begin{cases} e^{-\frac{1}{(1-\|x\|^2)}}, & \|x\| < 1, \\ \\ 0, & \|x\| \ge 1, \end{cases}$$

where $\|x\|$ denotes the Euclidean norm in $\mathbb{R}^n$.

- **Compact support:** By definition, $f(x) = 0$ for all $\|x\| \ge 1$, hence

  $$\text{supp}(f) \subseteq \overline{\{x \in \mathbb{R}^n : \|x\| < 1\}} = \{x \in \mathbb{R}^n : \|x\| \le 1\},$$

  which is a compact set. Therefore, f has compact support.

- **Smoothness For $\|x\| < 1$:** We have $f(x) = e^{-1/(1-\|x\|^2)}$. Since the function $x \mapsto \|x\|^2$ is smooth and the function $t \mapsto -1/(1-t)$ is smooth for $t < 1$, their composition is smooth. Hence, f is smooth.

- **Smoothness at $\|x\| = 1$:** Let $r = \|x\|$. Then $f(x) = e^{-1/(1-r^2)}$ for $r < 1$. As $r \to 1^-$, $f(x) \to 0$ faster than any polynomial in $(1 - r^2)$. Moreover, for any $k \ge 0$,

  $$\lim_{r\to 1^-} \frac{d^k}{dr^k} f(r) = 0.$$

  Hence all partial derivatives of f of any order extend continuously to 0 for $\|x\| \ge 1$. Therefore, $f \in C^\infty(\mathbb{R}^n)$.

Now, define $f_j(x) = \frac{f(jx)}{j}$. For this, we have

$$\|f_j\|_\infty = \max_{x\in\mathbb{R}^n} |f_j(x)| = \max_{x\in\mathbb{R}^n} \frac{1}{j}|f(jx)| = M\frac{1}{j} \to 0 \quad \text{as } j \to \infty$$

where $M = \max_{x\in\mathbb{R}^n} |f(jx)| = \max_{x\in\mathbb{R}^n} |f(x)| > 0$.

On the other hand by the chain rule,

$$Df_j(x) = Df(jx),$$

so that

$$\|Df_j\|_\infty = \max_{x\in\mathbb{R}^n} |Df_j(x)| = \max_{x\in\mathbb{R}^n} |Df(jx)| = \max_{x\in\mathbb{R}^n} |Df(x)| =: M^* > 0,$$

independent of j. Thus,

$$\|Df_j\|_\infty \not\to 0 \quad \text{as } j \to \infty$$

Therefore, D is not continuous.

Exercise 1.3

Verify the axioms of metric space:

(1) Non-negativity & Definiteness:
Since $\|\cdot\|$ is a norm, we have $\|x-y\| \geq 0$ for all $x, y \in E$. Hence, $d(x,y) \geq 0$. Also,

$$d(x,y) = \|x-y\| = 0 \iff x-y=0 \iff x=y.$$

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(2) Symmetry:

$$d(x,y) = \|x-y\| = \|-(y-x)\| = \|y-x\| = d(y,x)$$

for all $x, y \in E$.

(3) Triangle Inequality:
Using the triangle inequality of norm, we get

$$d(x,z) = \|x-z\| = \|(x-y)+(y-z)\| \leq \|x-y\| + \|y-z\| = d(x,y) + d(y,z)$$

for all $x, y, z \in E$.

Hence, $d(x,y) = \|x-y\|$ defines a metric on E.

Exercise 1.4

Let $x, y \in E$. Using the triangle inequality for a norm, we have

$$\|x\| = \|(x-y)+y\| \leq \|x-y\| + \|y\|,$$

which implies

$$\|x\| - \|y\| \leq \|x-y\|.$$

Similarly, swapping x and y, we get

$$\|y\| - \|x\| \leq \|x-y\|.$$

Combining the two inequalities, we obtain

$$-\|x-y\| \leq \|x\| - \|y\| \leq \|x-y\|,$$

or equivalently,

$$|\|x\| - \|y\|| \leq \|x-y\|.$$