

E. The Hahn-Banach Theorem

This Appendix contains several technical results, that are extremely useful in Functional Analysis. The following terminology is useful in formulating the statements.

DEFINITIONS. Let \mathbb{K} be one of the fields \mathbb{R} or \mathbb{C} , and let \mathcal{X} be a \mathbb{K} -vector space.

A. A map $q : \mathcal{X} \rightarrow \mathbb{R}$ is said to be a *quasi-seminorm*, if

- (i) $q(x + y) \leq q(x) + q(y)$, for all $x, y \in \mathcal{X}$;
- (ii) $q(tx) = tq(x)$, for all $x \in \mathcal{X}$ and all $t \in \mathbb{R}$ with $t \geq 0$.

B. A map $q : \mathcal{X} \rightarrow \mathbb{R}$ is said to be a *seminorm* if, in addition to the above two properties, it satisfies:

- (ii') $q(\lambda x) = |\lambda|q(x)$, for all $x \in \mathcal{X}$ and all $\lambda \in \mathbb{K}$.

Remark that if $q : \mathcal{X} \rightarrow \mathbb{R}$ is a seminorm, then $q(x) \geq 0$, for all $x \in \mathcal{X}$. (Use $2q(x) = q(x) + q(-x) \geq q(0) = 0$.)

There are several versions of the Hahn-Banach Theorem.

THEOREM E.1 (Hahn-Banach, \mathbb{R} -version). *Let \mathcal{X} be an \mathbb{R} -vector space. Suppose $q : \mathcal{X} \rightarrow \mathbb{R}$ is a quasi-seminorm. Suppose also we are given a linear subspace $\mathcal{Y} \subset \mathcal{X}$ and a linear map $\phi : \mathcal{Y} \rightarrow \mathbb{R}$, such that*

$$\phi(y) \leq q(y), \text{ for all } y \in \mathcal{Y}.$$

Then there exists a linear map $\psi : \mathcal{X} \rightarrow \mathbb{R}$ such that

- (i) $\psi|_{\mathcal{Y}} = \phi$;
- (ii) $\psi(x) \leq q(x)$ for all $x \in \mathcal{X}$.

PROOF. We first prove the Theorem in the following:

Particular Case: Assume $\dim \mathcal{X}/\mathcal{Y} = 1$.

This means there exists some vector $x_0 \in \mathcal{X}$ such that

$$\mathcal{X} = \{y + sx_0 : y \in \mathcal{Y}, s \in \mathbb{R}\}.$$

What we need is to prescribe the value $\psi(x_0)$. In other words, we need a number $\alpha \in \mathbb{R}$ such that, if we define $\psi : \mathcal{X} \rightarrow \mathbb{R}$ by $\psi(y + sx_0) = \phi(y) + s\alpha$, $\forall y \in \mathcal{Y}, s \in \mathbb{R}$, then this map satisfies condition (ii). For $s > 0$, condition (ii) reads:

$$\phi(y) + s\alpha \leq q(y + sx_0), \forall y \in \mathcal{Y}, s > 0,$$

and, upon dividing by s (set $z = s^{-1}y$), is equivalent to:

$$(1) \quad \alpha \leq q(z + x_0) - \phi(z), \forall z \in \mathcal{Y}.$$

For $s < 0$, condition (ii) reads (use $t = -s$):

$$\phi(y) - t\alpha \leq q(y - tx_0), \forall y \in \mathcal{Y}, t > 0,$$

and, upon dividing by t (set $w = t^{-1}y$), is equivalent to:

$$(2) \quad \alpha \geq \phi(w) - q(w - x_0), \forall w \in \mathcal{Y}.$$

Consider the sets

$$\begin{aligned} Z &= \{q(z + x_0) - \phi(z); z \in \mathcal{Y}\} \subset \mathbb{R} \\ W &= \{\phi(w) - q(w - x_0) : w \in \mathcal{Y}\} \subset \mathbb{R}. \end{aligned}$$

The conditions (1) and (2) are equivalent to the inequalities

$$(3) \quad \sup W \leq \alpha \leq \inf Z.$$

This means that, in order to find a real number α with the desired property, it suffices to prove that $\sup W \leq \inf Z$, which in turn is equivalent to

$$(4) \quad \phi(w) - q(w - x_0) \leq q(z + x_0) - \phi(z), \quad \forall z, w \in \mathcal{Y}.$$

But the condition (4) is equivalent to

$$\phi(z + w) \leq q(z + x_0) + q(w - x_0),$$

which is obviously satisfied because

$$\phi(z + w) \leq q(z + w) = q((z + x_0) + (w - x_0)) \leq q(z + x_0) + q(w - x_0).$$

Having proved the Theorem in this particular case, let us proceed now with the general case. Let us consider the set Ξ of all pairs (\mathcal{Z}, ν) with

- \mathcal{Z} is a subspace of \mathcal{X} such that $\mathcal{Z} \supset \mathcal{Y}$;
- $\nu : \mathcal{Z} \rightarrow \mathbb{R}$ is a linear functional such that
 - (i) $\nu|_{\mathcal{Y}} = \phi$;
 - (ii) $\nu(z) \leq q(z)$, for all $z \in \mathcal{Z}$.

Put an order relation \succ on Ξ as follows:

$$(\mathcal{Z}_1, \nu_1) \succ (\mathcal{Z}_2, \nu_2) \Leftrightarrow \begin{cases} \mathcal{Z}_1 \supset \mathcal{Z}_2 \\ \nu_1|_{\mathcal{Z}_2} = \nu_2 \end{cases}$$

Using Zorn's Lemma, Ξ possesses a maximal element (\mathcal{Z}, ψ) . The proof of the Theorem is finished once we prove that $\mathcal{Z} = \mathcal{X}$. Assume $\mathcal{Z} \subsetneq \mathcal{X}$ and choose a vector $x_0 \in \mathcal{X} \setminus \mathcal{Z}$. Form the subspace $\mathcal{V} = \{z + tx_0 : z \in \mathcal{Z}, t \in \mathbb{R}\}$ and apply the particular case of the Theorem for the inclusion $\mathcal{Z} \subset \mathcal{V}$, for $\psi : \mathcal{Z} \rightarrow \mathbb{R}$ and for the quasi-seminorm $q|_{\mathcal{V}} : \mathcal{V} \rightarrow \mathbb{R}$. It follows that there exists some linear functional $\eta : \mathcal{M} \rightarrow \mathbb{R}$ such that

- (i) $\eta|_{\mathcal{Z}} = \psi$ (in particular we will also have $\eta|_{\mathcal{Y}} = \phi$);
- (ii) $\eta(v) \leq q(v)$, for all $v \in \mathcal{V}$.

But then the element $(\mathcal{V}, \eta) \in \Xi$ will contradict the maximality of (\mathcal{Z}, ψ) . \square

THEOREM E.2 (Hahn-Banach, \mathbb{C} -version). *Let \mathcal{X} be an \mathbb{C} -vector space. Suppose $q : \mathcal{X} \rightarrow \mathbb{R}$ is a quasi-seminorm. Suppose also we are given a linear subspace $\mathcal{Y} \subset \mathcal{X}$ and a linear map $\phi : \mathcal{Y} \rightarrow \mathbb{C}$, such that*

$$\operatorname{Re} \phi(y) \leq q(y), \text{ for all } y \in \mathcal{Y}.$$

Then there exists a linear map $\psi : \mathcal{X} \rightarrow \mathbb{R}$ such that

- (i) $\psi|_{\mathcal{Y}} = \phi$;
- (ii) $\operatorname{Re} \psi(x) \leq q(x)$ for all $x \in \mathcal{X}$.

PROOF. Regard for the moment both \mathcal{X} and \mathcal{Y} as \mathbb{R} -vector spaces. Define the \mathbb{R} -linear map $\phi_1 : \mathcal{Y} \rightarrow \mathbb{R}$ by $\phi_1(y) = \operatorname{Re} \phi(y)$, for all $y \in \mathcal{Y}$, so that we have

$$\phi_1(y) \leq q(y), \quad \forall y \in \mathcal{Y}.$$

Use Theorem E.1 to find an \mathbb{R} -linear map $\psi_1 : \mathcal{X} \rightarrow \mathbb{R}$ such that

- (i) $\psi_1|_{\mathcal{Y}} = \phi_1$;
- (ii) $\psi_1(x) \leq q(x)$, for all $x \in \mathcal{X}$.

Define the map $\psi : \mathcal{X} \rightarrow \mathbb{C}$ by

$$\psi(x) = \psi_1(x) - i\psi_1(ix), \text{ for all } x \in \mathcal{X}.$$

Claim 1: ψ is \mathbb{C} -linear.

It is obvious that ψ is \mathbb{R} -linear, so the only thing to prove is that $\psi(ix) = i\psi(x)$, for all $x \in \mathcal{X}$. But this is quite obvious:

$$\begin{aligned} \psi(ix) &= \psi_1(ix) - i\psi_1(i^2x) = \psi_1(ix) - i\psi_1(-x) = \\ &= -i^2\psi_1(x) + i\psi_1(x) = i(\psi_1(x) - i\psi_1(ix)) = i\psi(x), \quad \forall x \in \mathcal{X}. \end{aligned}$$

Because of the way ψ is defined, and because ψ_1 is real-valued, condition (ii) in the Theorem follows immediately

$$\operatorname{Re} \psi(x) = \psi_1(x) \leq q(x), \quad \forall x \in \mathcal{X},$$

so in order to finish the proof, we need to prove condition (i) in the Theorem, (i.e. $\psi|_{\mathcal{Y}} = \phi$). This follows from the fact that $\phi_1 = \psi_1|_{\mathcal{Y}}$, and from:

Claim 2: For every $y \in \mathcal{Y}$, we have $\phi(y) = \phi_1(y) - i\phi_1(iy)$.

But this is quite obvious, because

$$\operatorname{Im} \phi(y) = -\operatorname{Re}(i\phi(y)) = -\operatorname{Re} \phi(iy) = -\phi_1(iy), \quad \forall y \in \mathcal{Y}.$$

□

THEOREM E.3 (Hahn-Banach, for seminorms). *Let \mathcal{X} be a \mathbb{K} -vector space (\mathbb{K} is either \mathbb{R} or \mathbb{C}). Suppose q is a seminorm on \mathcal{X} . Suppose also we are given a linear subspace $\mathcal{Y} \subset \mathcal{X}$ and a linear map $\phi : \mathcal{Y} \rightarrow \mathbb{K}$, such that*

$$|\phi(y)| \leq q(y), \text{ for all } y \in \mathcal{Y}.$$

Then there exists a linear map $\psi : \mathcal{X} \rightarrow \mathbb{K}$ such that

- (i) $\psi|_{\mathcal{Y}} = \phi$;
- (ii) $|\psi(x)| \leq q(x)$ for all $x \in \mathcal{X}$.

PROOF. We are going to apply Theorems E.1 and E.2, using the fact that q is also a quasi-seminorm.

THE CASE $\mathbb{K} = \mathbb{R}$. Remark that

$$\phi(y) \leq |\phi(y)| \leq q(y), \quad \forall y \in \mathcal{Y}.$$

So we can apply Theorem E.1 and find $\psi : \mathcal{X} \rightarrow \mathbb{R}$ with

- (i) $\psi|_{\mathcal{Y}} = \phi$;
- (ii) $\psi(x) \leq q(x)$, for all $x \in \mathcal{X}$.

Using condition (ii) we also get

$$-\psi(x) = \psi(-x) \leq q(-x) = q(x), \text{ for all } x \in \mathcal{X}.$$

In other words we get

$$\pm\psi(x) \leq q(x), \text{ for all } x \in \mathcal{X},$$

which of course gives the desired property (ii) in the Theorem.

THE CASE $\mathbb{K} = \mathbb{C}$. Remark that

$$\operatorname{Re} \phi(y) \leq |\phi(y)| \leq q(y), \quad \forall y \in \mathcal{Y}.$$

So we can apply Theorem E.2 and find $\psi : \mathcal{X} \rightarrow \mathbb{R}$ with

- (i) $\psi|_{\mathcal{Y}} = \phi$;
- (ii) $\operatorname{Re} \psi(x) \leq q(x)$, for all $x \in \mathcal{X}$.

Using condition (ii) we also get

$$(5) \quad \operatorname{Re}(\lambda\psi(x)) = \operatorname{Re} \psi(\lambda x) \leq q(\lambda x) = q(x), \text{ for all } x \in \mathcal{X} \text{ and all } \lambda \in \mathbb{T}.$$

(Here $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.) Fix for the moment $x \in \mathcal{X}$. There exists some $\lambda \in \mathbb{T}$ such that $|\psi(x)| = \lambda\psi(x)$. For this particular λ we will have $\operatorname{Re}(\lambda\psi(x)) = |\psi(x)|$, so the inequality (5) will give

$$|\psi(x)| \leq q(x).$$

□

In the remainder of this section we will discuss the geometric form of the Hahn-Banach theorems. We begin by describing a method of constructing quasi-seminorms.

PROPOSITION E.1. *Let \mathcal{X} be a real vector space. Suppose $\mathcal{C} \subset \mathcal{X}$ is a convex subset, which contains 0, and has the property*

$$(6) \quad \bigcup_{t>0} t\mathcal{C} = \mathcal{X}.$$

For every $x \in \mathcal{X}$ we define

$$Q_{\mathcal{C}}(x) = \inf\{t > 0 : x \in t\mathcal{C}\}.$$

(By (6) the set in the right hand side is non-empty.) Then the map $Q_{\mathcal{C}} : \mathcal{X} \rightarrow \mathbb{R}$ is a quasi-seminorm.

PROOF. For every $x \in \mathcal{X}$, let us define the set

$$T_{\mathcal{C}}(x) = \{t > 0 : x \in t\mathcal{C}\}.$$

It is pretty clear that, since $0 \in \mathcal{C}$, we have

$$T_{\mathcal{C}}(0) = (0, \infty),$$

so we get

$$Q_{\mathcal{C}}(0) = \inf T_{\mathcal{C}}(0) = 0.$$

Claim 1: For every $x \in \mathcal{X}$ and every $\lambda > 0$, one has the equality

$$T_{\mathcal{C}}(\lambda x) = \lambda T_{\mathcal{C}}(x).$$

Indeed, if $t \in T_{\mathcal{C}}(\lambda x)$, we have $\lambda x \in t\mathcal{C}$, which means that $\lambda^{-1}tx \in \mathcal{C}$, i.e. $\lambda^{-1}t \in T_{\mathcal{C}}(x)$. Consequently we have

$$t = \lambda(\lambda^{-1}t) \in \lambda T_{\mathcal{C}}(x),$$

which proves the inclusion

$$T_{\mathcal{C}}(\lambda x) \subset \lambda T_{\mathcal{C}}(x).$$

To prove the other inclusion, we start with some $s \in \lambda T_{\mathcal{C}}(x)$, which means that there exists some $t \in T_{\mathcal{C}}(x)$ with $\lambda t = s$. The fact that $t = \lambda^{-1}s$ belongs to $T_{\mathcal{C}}(x)$ means that $x \in \lambda^{-1}s\mathcal{C}$, so get $\lambda x \in s\mathcal{C}$, so s indeed belongs to $T_{\mathcal{C}}(\lambda x)$.

Claim 2: For every $x, y \in \mathcal{X}$, one has the inclusion¹

$$T_{\mathcal{C}}(x + y) \supset T_{\mathcal{C}}(x) + T_{\mathcal{C}}(y).$$

¹For subsets $T, S \subset \mathbb{R}$ we define $T + S = \{t + s : t \in T, s \in S\}$.

Start with some $t \in T_{\mathcal{C}}(x)$ and some $s \in T_{\mathcal{C}}(y)$. Define the elements $u = t^{-1}x$ and $v = s^{-1}y$. Since $u, v \in \mathcal{C}$, and \mathcal{C} is convex, it follows that \mathcal{C} contains the element

$$\frac{t}{t+s}u + \frac{s}{t+s}v = \frac{1}{t+s}(x+y),$$

which means that $x+y \in (t+s)\mathcal{C}$, so $t+s$ indeed belongs to $T_{\mathcal{C}}(x+y)$.

We can now conclude the proof. If $x \in \mathcal{X}$ and $\lambda > 0$, then the equality

$$Q_{\mathcal{C}}(\lambda x) = \lambda Q_{\mathcal{C}}(x)$$

is an immediate consequence of Claim 1. If $x, y \in \mathcal{X}$, then the inequality

$$Q_{\mathcal{C}}(x+y) \leq \lambda Q_{\mathcal{C}}(x) + Q_{\mathcal{C}}(y)$$

is an immediate consequence of Claim 2. \square

DEFINITION. Under the hypothesis of the above proposition, the quasi-seminorm $Q_{\mathcal{C}}$ is called the *Minkowski functional* associated with the set \mathcal{C} .

REMARK E.1. Let \mathcal{X} be a real vector space. Suppose $\mathcal{C} \subset \mathcal{X}$ is a convex subset, which contains 0, and has the property (6). Then one has the inclusions

$$\{x \in \mathcal{X} : Q_{\mathcal{C}}(x) < 1\} \subset \mathcal{C} \subset \{x \in \mathcal{X} : Q_{\mathcal{C}}(x) \leq 1\}.$$

The second inclusion is pretty obvious, since if we start with some $x \in \mathcal{C}$, using the notations from the proof of Proposition E.1, we have $1 \in T_{\mathcal{C}}(x)$, so

$$Q_{\mathcal{C}}(x) = \inf T_{\mathcal{C}}(x) \leq 1.$$

To prove the first inclusion, start with some $x \in \mathcal{X}$ with $Q_{\mathcal{C}}(x) < 1$. In particular this means that there exists some $t \in (0, 1)$ such that $x \in t\mathcal{C}$. Define the vector $y = t^{-1}x \in \mathcal{C}$ and notice now that, since \mathcal{C} is convex, it will contain the convex combination $ty + (1-t)0 = x$.

DEFINITION. A *topological vector space* is a vector space \mathcal{X} over \mathbb{K} (which is either \mathbb{R} or \mathbb{C}), which is also a topological space, such that the maps

$$\mathcal{X} \times \mathcal{X} \ni (x, y) \longmapsto x + y \in \mathcal{X}$$

$$\mathbb{K} \times \mathcal{X} \ni (\lambda, x) \longmapsto \lambda x \in \mathcal{X}$$

are continuous.

REMARK E.2. Let \mathcal{X} be a real topological vector space. Suppose $\mathcal{C} \subset \mathcal{X}$ is a convex *open* subset, which contains 0. Then \mathcal{C} has the property (6). Moreover (compare with Remark E.1), one has the equality

$$(7) \quad \{x \in \mathcal{X} : Q_{\mathcal{C}}(x) < 1\} = \mathcal{C}.$$

To prove this remark, we define for each $x \in \mathcal{X}$, the function

$$F_x : \mathbb{R} \ni t \longmapsto tx \in \mathcal{X}.$$

Since \mathcal{X} is a topological vector space, the map F_x , $x \in \mathcal{X}$ are continuous. To prove the property (6) we start with an arbitrary $x \in \mathcal{X}$, and we use the continuity of the map F_x at 0. Since \mathcal{C} is a neighborhood of 0, there exists some $\rho > 0$ such that

$$F_x(t) \in \mathcal{C}, \quad \forall t \in [-\rho, \rho].$$

In particular we get $\rho x \in \mathcal{C}$, which means that $x \in \rho^{-1}\mathcal{C}$.

To prove the equality (7) we only need to prove the inclusion “ \supset ” (since the inclusion “ \subset ” holds in general, by Remark E.1). Start with some element $x \in \mathcal{C}$.

Using the continuity of the map F_x at 1, plus the fact that $F_x(1) = x \in \mathcal{C}$, there exists some $\varepsilon > 0$, such that

$$F_x(t) \in \mathcal{C}, \quad \forall t \in [1 - \varepsilon, 1 + \varepsilon].$$

In particular, we have $F(1 + \varepsilon) \in \mathcal{C}$, which means precisely that

$$x \in (1 + \varepsilon)^{-1}\mathcal{C}.$$

This gives the inequality

$$Q_{\mathcal{C}}(x) \leq (1 + \varepsilon)^{-1},$$

so we indeed get $Q_{\mathcal{C}}(x) < 1$.

The first geometric version of the Hahn-Banach Theorem is:

LEMMA E.1. *Let \mathcal{X} be a real topological vector space, and let $\mathcal{C} \subset \mathcal{X}$ be a convex open set which contains 0. If $x_0 \in \mathcal{X}$ is some point which does not belong to \mathcal{C} , then there exists a linear continuous map $\phi : \mathcal{X} \rightarrow \mathbb{R}$, such that*

- $\phi(x_0) = 1$;
- $\phi(v) < 1, \forall v \in \mathcal{C}$.

PROOF. Consider the linear subspace

$$\mathcal{Y} = \mathbb{R}x_0 = \{tx_0 : t \in \mathbb{R}\},$$

and define $\psi : \mathcal{Y} \rightarrow \mathbb{R}$ by

$$\psi(tx_0) = t, \quad \forall t \in \mathbb{R}.$$

It is obvious that ψ is linear, and $\psi(x_0) = 1$.

Claim: One has the inequality

$$\psi(y) \leq Q_{\mathcal{C}}(y), \quad \forall y \in \mathcal{Y}.$$

Let y be represented as $y = tx_0$ for some $t \in \mathbb{R}$. If $t \leq 0$, the inequality is clear, because $\psi(y) = t \leq 0$ and the right hand side $Q_{\mathcal{C}}(y)$ is always non-negative. Assume $t > 0$. Since $Q_{\mathcal{C}}$ is a quasi-seminorm, we have

$$(8) \quad Q_{\mathcal{C}}(y) = Q_{\mathcal{C}}(tx_0) = tQ_{\mathcal{C}}(x_0),$$

and the fact that $x_0 \notin \mathcal{C}$ will give (by Remark E.2) the inequality $Q_{\mathcal{C}}(x_0) \geq 1$. Since $t > 0$, the computation (8) can be continued with

$$Q_{\mathcal{C}}(y) = tQ_{\mathcal{C}}(x_0) \geq t = \psi(y),$$

so the Claim follows also in this case.

Use now the Hahn-Banach Theorem, to find a linear map $\phi : \mathcal{X} \rightarrow \mathbb{R}$ such that

- (i) $\phi|_{\mathcal{Y}} = \psi$;
- (ii) $\phi(x) \leq Q_{\mathcal{C}}(x), \forall x \in \mathcal{X}$.

It is obvious that (i) gives $\phi(x_0) = \psi(x_0) = 1$. If $v \in \mathcal{C}$, then by Remark E.2 we have $Q_{\mathcal{C}}(v) < 1$, so by (ii) we also get $\phi(v) < 1$. This means that the only thing that remains to be proven is the continuity of ϕ . Since ϕ is linear, we only need to prove that ϕ is continuous at 0. Start with some $\varepsilon > 0$. We must find some open set $\mathcal{U}_{\varepsilon} \subset \mathcal{X}$, with $\mathcal{U}_{\varepsilon} \ni 0$, such that

$$|\phi(u)| < \varepsilon, \quad \forall u \in \mathcal{U}_{\varepsilon}.$$

We take $\mathcal{U}_{\varepsilon} = (\varepsilon\mathcal{C}) \cap (-\varepsilon\mathcal{C})$. Notice that, for every $u \in \mathcal{U}_{\varepsilon}$, we have $\pm u \in \varepsilon\mathcal{C}$, which gives $\varepsilon^{-1}(\pm u) \in \mathcal{C}$. By Remark E.2 this gives $Q_{\mathcal{C}}(\varepsilon^{-1}(\pm u)) < 1$, which gives

$$Q_{\mathcal{C}}(\pm u) < \varepsilon.$$

Then using property (ii) we immediately get

$$\phi(\pm u) < \varepsilon,$$

and we are done. \square

It turns out that the above result is a particular case of a more general result:

THEOREM E.4 (Hahn-Banach Separation Theorem - real case). *Let \mathcal{X} be a real topological vector space, let $\mathcal{A}, \mathcal{B} \subset \mathcal{X}$ be non-empty convex sets with \mathcal{A} open, and $\mathcal{A} \cap \mathcal{B} = \emptyset$. Then there exists a linear continuous map $\phi : \mathcal{X} \rightarrow \mathbb{R}$, and a real number α , such that*

$$\phi(a) < \alpha \leq \phi(b), \quad \forall a \in \mathcal{A}, b \in \mathcal{B}.$$

PROOF. Fix some points $a_0 \in \mathcal{A}$, $b_0 \in \mathcal{B}$, and define the set

$$\mathcal{C} = \mathcal{A} - \mathcal{B} + b_0 - a_0 = \{a - b + b_0 - a_0 : a \in \mathcal{A}, b \in \mathcal{B}\}.$$

It is straightforward that \mathcal{C} is convex and contains 0. The equality

$$\mathcal{C} = \bigcup_{b \in \mathcal{B}} (\mathcal{A} + b_0 - a_0)$$

shows that \mathcal{C} is also open. Define the vector $x_0 = b_0 - a_0$. Since $\mathcal{A} \cap \mathcal{B} = \emptyset$, it is clear that $x_0 \notin \mathcal{C}$.

Use Lemma E.1 to produce a linear continuous map $\phi : \mathcal{X} \rightarrow \mathbb{R}$ such that

- (i) $\phi(x_0) = 1$;
- (ii) $\phi(v) < 1, \forall v \in \mathcal{C}$.

By the definition of x_0 and \mathcal{C} , we have $\phi(b_0) = \phi(a_0) + 1$, and

$$\phi(a) < \phi(b) + \phi(a_0) - \phi(b_0) + 1, \quad \forall a \in \mathcal{A}, b \in \mathcal{B},$$

which gives

$$(9) \quad \phi(a) < \phi(b), \quad \forall a \in \mathcal{A}, b \in \mathcal{B}.$$

Put

$$\alpha = \inf_{b \in \mathcal{B}} \phi(b).$$

The inequalities (9) give

$$(10) \quad \phi(a) \leq \alpha \leq \phi(b), \quad \forall a \in \mathcal{A}, b \in \mathcal{B}.$$

The proof will be complete once we prove the following

Claim: One has the inequality

$$\phi(a) < \alpha, \quad \forall a \in \mathcal{A}.$$

Suppose the contrary, i.e. there exists some $a_1 \in \mathcal{A}$ with $\phi(a_1) = \alpha$. Using the continuity of the map

$$\mathbb{R} \ni t \mapsto a_1 + tx_0 \in \mathcal{X}$$

there exists some $\varepsilon > 0$ such that

$$a_1 + tx_0 \in \mathcal{A}, \quad \forall t \in [-\varepsilon, \varepsilon].$$

In particular, by (10) one has

$$\phi(a_1 + \varepsilon x_0) \leq \alpha,$$

which means that

$$\alpha + \varepsilon \leq \alpha,$$

which is clearly impossible. \square

THEOREM E.5 (Hahn-Banach Separation Theorem - complex case). *Let \mathcal{X} be a complex topological vector space, let $\mathcal{A}, \mathcal{B} \subset \mathcal{X}$ be non-empty convex sets with \mathcal{A} open, and $\mathcal{A} \cap \mathcal{B} = \emptyset$. Then there exists a linear continuous map $\phi : \mathcal{X} \rightarrow \mathbb{C}$, and a real number α , such that*

$$\operatorname{Re} \phi(a) < \alpha \leq \operatorname{Im} \phi(b), \quad \forall a \in \mathcal{A}, b \in \mathcal{B}.$$

PROOF. Regard \mathcal{X} as a real topological vector space, and apply the real version to produce an \mathbb{R} -linear continuous map $\phi_1 : \mathcal{X} \rightarrow \mathbb{R}$, and a real number α , such that

$$\phi_1(a) < \alpha \leq \phi_1(b), \quad \forall a \in \mathcal{A}, b \in \mathcal{B}.$$

Then the function $\phi : \mathcal{X} \rightarrow \mathbb{C}$ defined by

$$\phi(x) = \phi_1(x) - i\phi_1(ix), \quad x \in \mathcal{X}$$

will clearly satisfy the desired properties. \square

There is another version of the Hahn-Banach Separation Theorem, which holds for a special type of topological vector spaces. Before we discuss these, we shall need a technical result.

LEMMA E.2. *Let \mathcal{X} be a topological vector space, let $\mathcal{C} \subset \mathcal{X}$ be a compact set, and let $\mathcal{D} \subset \mathcal{X}$ be a closed set. Then the set*

$$\mathcal{C} + \mathcal{D} = \{x + y : x \in \mathcal{C}, y \in \mathcal{D}\}$$

is closed.

PROOF. Start with some point $p \in \overline{\mathcal{C} + \mathcal{D}}$, and let us prove that $p \in \mathcal{C} + \mathcal{D}$. For every neighborhood \mathcal{U} of 0, the set $p + \mathcal{U}$ is a neighborhood of p , so by assumption, we have

$$(11) \quad (p + \mathcal{U}) \cap (\mathcal{C} + \mathcal{D}) \neq \emptyset.$$

Define, for each neighborhood \mathcal{U} of 0, the set

$$\mathcal{A}_{\mathcal{U}} = (p + \mathcal{U} - \mathcal{D}) \cap \mathcal{C}.$$

Using (11), it is clear that $\mathcal{A}_{\mathcal{U}}$ is non-empty. It is also clear that, if $\mathcal{U}_1 \subset \mathcal{U}_2$, then $\mathcal{A}_{\mathcal{U}_1} \subset \mathcal{A}_{\mathcal{U}_2}$. Using the compactness of \mathcal{C} , it follows that

$$\bigcap_{\substack{\mathcal{U} \text{ neighborhood} \\ \text{of } 0}} \overline{\mathcal{A}_{\mathcal{U}}} \neq \emptyset.$$

Choose then a point q in the above intersection. It follows that

$$(q + \mathcal{V}) \cap \mathcal{A}_{\mathcal{U}} \neq \emptyset,$$

for any two neighborhoods \mathcal{U} and \mathcal{V} of 0. In other words, for any two such neighborhoods of 0, we have

$$(12) \quad (q + \mathcal{V} - \mathcal{U}) \cap (p - \mathcal{D}) \neq \emptyset.$$

Fix now an arbitrary neighborhood \mathcal{W} of 0. Using the continuity of the map

$$\mathcal{X} \times \mathcal{X} \ni (x_1, x_2) \mapsto x_1 - x_2 \in \mathcal{X},$$

there exist neighborhoods \mathcal{U} and \mathcal{V} of 0, such that $\mathcal{U} - \mathcal{V} \subset \mathcal{W}$. Then $q + \mathcal{V} - \mathcal{U} \subset q - \mathcal{W}$, so (12) gives

$$(q - \mathcal{W}) \cap (p - \mathcal{D}) \neq \emptyset,$$

which yields

$$(p - q + \mathcal{W}) \cap \mathcal{D} \neq \emptyset.$$

Since this is true for all neighborhoods \mathcal{W} of 0, we get $p - q \in \overline{\mathcal{D}}$, and since \mathcal{D} is closed, we finally get $p - q \in \mathcal{D}$. Since, by construction we have $q \in \mathcal{C}$, it follows that the point $p = q + (p - q)$ indeed belongs to $\mathcal{C} + \mathcal{D}$. \square

DEFINITION. A topological vector space \mathcal{X} is said to be *locally convex*, if every point has a fundamental system of convex open neighborhoods. This means that for every $x \in \mathcal{X}$ and every neighborhood N of x , there exists a convex open set D , with $x \in D \subset N$.

THEOREM E.6 (Hahn-Banach Separation Theorem for Locally Convex Spaces).
Let \mathbb{K} be one of the fields \mathbb{R} or \mathbb{C} , and let \mathcal{X} be a locally convex \mathbb{K} -vector space. Suppose $\mathcal{C}, \mathcal{D} \subset \mathcal{X}$ are convex sets, with \mathcal{C} compact, \mathcal{D} closed, and $\mathcal{C} \cap \mathcal{D} = \emptyset$. Then there exists a linear continuous map $\phi : \mathcal{X} \rightarrow \mathbb{K}$, and two numbers $\alpha, \beta \in \mathbb{R}$, such that

$$\operatorname{Re} \phi(x) \leq \alpha < \beta \leq \operatorname{Re} \phi(y), \quad \forall x \in \mathcal{C}, y \in \mathcal{D}.$$

PROOF. Consider the convex set $\mathcal{B} = \mathcal{D} - \mathcal{C}$. By Lemma ??, \mathcal{B} is closed. Since $\mathcal{C} \cap \mathcal{D} = \emptyset$, we have $0 \notin \mathcal{B}$. Since \mathcal{B} is closed, its complement $\mathcal{X} \setminus \mathcal{B}$ will then be a neighborhood of 0. Since \mathcal{X} is locally convex, there exists a convex open set \mathcal{A} , with $0 \in \mathcal{A} \subset \mathcal{X} \setminus \mathcal{B}$. In particular we have $\mathcal{A} \cap \mathcal{B} = \emptyset$. Applying the suitable version of the Hahn-Banach Theorem (real or complex case), we find a linear continuous map $\phi : \mathcal{X} \rightarrow \mathbb{K}$, and a real number ρ , such that

$$\operatorname{Re} \phi(a) < \rho \leq \operatorname{Re} \phi(b), \quad \forall a \in \mathcal{A}, b \in \mathcal{B}.$$

Notice that, since $\mathcal{A} \ni 0$, we get $\rho > 0$. Then the inequality

$$\rho \leq \operatorname{Re} \phi(b), \quad b \in \mathcal{B}$$

gives

$$\operatorname{Re} \phi(y) - \operatorname{Re} \phi(x) \geq \rho > 0, \quad \forall x \in \mathcal{C}, y \in \mathcal{D}.$$

Then if we define

$$\beta = \inf_{y \in \mathcal{D}} \operatorname{Re} \phi(y) \quad \text{and} \quad \alpha = \sup_{x \in \mathcal{C}} \operatorname{Re} \phi(x),$$

we get $\beta \geq \alpha + \rho$, and we are done. \square