Banach Spaces 2026 (MATS2220)

Assignment 1

Solutions

Exercise 1.1

Let X, Y be normed vector spaces and let $T_n, T : X \to Y$ be linear maps and

$$\|T_n - T\| \to 0.$$

Fix any $x \in X$. If $x = 0$, the result is trivial(as 0 maps to 0 under T).
Now suppose $x \neq 0$. Then we can write

$$x = \|x\| \cdot \frac{x}{\|x\|}.$$

Using linearity,

$$(T_n - T)(x) = \|x\| \cdot (T_n - T)\left(\frac{x}{\|x\|}\right).$$

By taking norms,

$$\|(T_n - T)(x)\| = \|x\| \cdot \left\|(T_n - T)\left(\frac{x}{\|x\|}\right)\right\|.$$

Since $\left\|\frac{x}{\|x\|}\right\| = 1$, we have

$$\left\|(T_n - T)\left(\frac{x}{\|x\|}\right)\right\| \leq \|T_n - T\|.$$

Thus,

$$\|(T_n - T)(x)\| \leq \|x\| \cdot \|T_n - T\|.$$

Because $\|T_n - T\| \to 0$, it follows that

$$\|(T_n - T)(x)\| \to 0.$$

Therefore,

$$T_n(x) \to T(x) \quad \forall x.$$

$\square$

Exercise 1.2

Let $X = \ell^2$, and define operators $T_n : \ell^2 \to \ell^2$ by

$$T_n(x_1, x_2, x_3, \dots) = (0, \dots, 0, x_n, 0, 0, \dots),$$

where x_n appears in the n-th coordinate. Let $T : \ell^2 \to \ell^2$ be the zero operator, $T(x) = 0$.

For $x = (x_1, x_2, \dots) \in \ell^2$. We have

$$T_n(x) = (0, \dots, x_n, 0, \dots).$$

Since $x \in \ell^2$, we have $x_n \to 0$. Hence,

$$\|T_n(x)\| = |x_n| \to 0 = \|T(x)\|.$$

Thus $T_n(x) \to T(x)$ for every $x \in \ell^2$, i.e. $T_n \to T$ pointwise.

Now, we will show that it does not converge in operator norm.

Observe that

$$\|T_n - T\| = \|T_n\|.$$

Let $e_n = (0, \dots, 1, 0, \dots)$. Then

$$T_n(e_n) = e_n,$$

so

$$\|T_n(e_n)\| = 1.$$

Hence,

$$\|T_n\| \geq 1 \quad \text{for all } n.$$

Thus,

$$\|T_n - T\| \not\to 0.$$

Therefore,

Pointwise convergence does not imply convergence in operator norm.

□

Exercise 1.3

consider the mapping $T : \ell^p \to \ell^p$, defined by

$$T(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots),$$

.

We want show that T is linear and preserves the ℓ^p-norm.

The Linearity is clear via component-wise by using the properties of real numbers.

Now let $x \in \ell^p$, then

Case $1 \leq p < \infty$**:**

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}.$$

Then

$$\|T(x)\|_p = (|0|^p + |x_1|^p + |x_2|^p + \cdots)^{1/p} = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} = \|x\|_p.$$

Case $p = \infty$:

$$\|x\|_\infty = \sup_n |x_n|.$$

Then

$$\|T(x)\|_\infty = \sup\{0, |x_1|, |x_2|, \dots\} = \sup_n |x_n| = \|x\|_\infty.$$

Therefore

$$\|T(x)\|_p = \|x\|_p \quad \text{for all } x \in \ell^p.$$

Hence, T is a linear isometry. □

Exercise 1.4

Recall that $T : \ell^p \to \ell^p$ is defined by

$$T(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots).$$

Define an operator $S : \ell^p \to \ell^p$ by

$$S(y_1, y_2, y_3, \dots) = (y_2, y_3, y_4, \dots).$$

Let $x = (x_1, x_2, x_3, \dots) \in \ell^p$. Then

$$T(x) = (0, x_1, x_2, x_3, \dots).$$

Applying S, we get

$$S(T(x)) = (x_1, x_2, x_3, \dots) = x.$$

Hence,

$$ST = I.$$

But T is not invertible as it is not surjective.
Consider $e_1 = (1, 0, 0, \dots) \in \ell^p$. Suppose there exists $x \in \ell^p$ such that $T(x) = e_1$. But

$$T(x) = (0, x_1, x_2, \dots),$$

whose first coordinate is always 0, so it can never equal e_1.
Therefore

$$T \text{ is not invertible.}$$

□

Exercise 1.5

Rewrite the equation as

$$f(y) = g(y) - h(y) \int_0^1 f(x)h(x)\,dx.$$

Let $X = C([0,1])$ with the norm $\|\cdot\|_\infty$, and define an operator $T : X \to X$ by

$$(Tf)(y) = g(y) - h(y) \int_0^1 f(x)h(x)\,dx.$$

We will check that T is a contraction.
Let $f_1, f_2 \in X$. Then

$$(Tf_1)(y) - (Tf_2)(y) = -h(y) \int_0^1 (f_1(x) - f_2(x))h(x)\,dx.$$

By the sup norm,

$$\|Tf_1 - Tf_2\|_\infty \leq \|h\|_\infty \left| \int_0^1 (f_1 - f_2)(x)h(x)\,dx \right|.$$

Using Cauchy–Schwarz inequality,

$$\left| \int_0^1 (f_1 - f_2)(x)h(x)\,dx \right| \leq \|f_1 - f_2\|_2 \|h\|_2.$$

Moreover, we know that on $[0,1]$ we have

$$\|f_1 - f_2\|_2 \leq \|f_1 - f_2\|_\infty \quad \text{and} \quad \|h\|_\infty \leq \|h\|_2.$$

Therefore,

$$\|Tf_1 - Tf_2\|_\infty \leq \|h\|_2^2 \|f_1 - f_2\|_\infty.$$

Because $\|h\|_2 < 1$, we have $\|h\|_2^2 < 1$, so T is a contraction.

Since $C([0,1])$ with $\|\cdot\|_\infty$ is a Banach space, Banach's fixed point theorem implies that T has a unique fixed point $f \in C([0,1])$.

This fixed point satisfies

$$f(y) = g(y) - h(y) \int_0^1 f(x)h(x)\,dx.$$

□

Exercise 1.6

Since $C([0,1]) \subset L^2([0,1])$. Consider the equation with $L^2([0,1])$ inner product

$$f(y) + h(y)\int_0^1 f(x)h(x)\,dx = g(y).$$

The equation can be written as

$$f + \langle f, h\rangle h = g.$$

with

$$\langle f, h\rangle = \int_0^1 f(x)h(x)\,dx,$$

Define the map

$$Tf = f + \langle f, h\rangle h.$$

We solve $Tf = g$ using an orthogonal decomposition. Write

$$L^2([0,1]) = \text{span}\{h\} \oplus h^{\perp},$$

and decompose

$$f = f_{\parallel} + f_{\perp}, \quad \text{where } f_{\parallel} = \alpha h,\ f_{\perp} \perp h.$$

Then

$$\langle f, h\rangle = \alpha\|h\|_2^2.$$

Applying T,

$$Tf = f_{\perp} + \alpha h + \alpha\|h\|_2^2 h = f_{\perp} + \alpha(1 + \|h\|_2^2)h.$$

Similarly decompose $g = g_{\perp} + g_{\parallel}$. Comparing components:

$$f_{\perp} = g_{\perp},$$
$$\alpha(1 + \|h\|_2^2)h = g_{\parallel}.$$

Since

$$g_{\parallel} = \frac{\langle g, h\rangle}{\|h\|_2^2} h,$$

we obtain

$$\alpha = \frac{\langle g, h\rangle}{\|h\|_2^2(1 + \|h\|_2^2)}.$$

Thus f is uniquely determined.

Note: Observe that this process is independent of the assumptiopn that whether $\|h\|_2 \geq 1$ or $\|h\|_2 \leq 1$.